Multiple View Reconstruction of a Quadric of Revolution from its Occluding Contours

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Abstract. The problem of reconstructing a quadric from its occluding contours is one of the earliest problems in computer vision e.g., see [1–3]. It is known that three contours from three views are required for this problem to be well-posed while Cross *et al.* have proved in [4] that, with only two contours, what can be obtained is a 1D linear family of solutions in the dual projective space.
In this work, we describe a multiple view algorithm that *unambiguously* reconstructs so-called Prolate Quadrics of Revolution (PQoR's, see text), given at least *two* finite projective cameras (see terminology in [5, p157]). In particular, we show how to obtain a closed-form solution.
The key result on which is based this work is a dual parameterization of a PQoR, using a 7-dof 'linear combination' of the quadric dual to the principal focus-pair and the Dual Absolute Quadric (DAQ).
One of the contributions is to prove that the images of the principal foci of a PQoR can be recovered set-wise from the images of the PQoR and the DAQ. The performance of the proposed algorithm is illustrated on simulations and experiments with real images.

1 Introduction

The now well-established maturity of 3D reconstruction paradigms in computer vision is due in part to the contribution of projective geometry, which allowed in particular to design stratified reconstruction strategies and to understand the link between camera calibration and the Euclidean structure of a projective 2- or 3- space [5]. More recently, projective geometry also threw light on some earlier vision problems like the 3D reconstruction of quadratic surfaces, quadratic curves or surfaces of revolution, from one or multiple views [4, 6–10].

The image of a general 9-dof quadric is a 5-dof conic which is usually referred to as the ***occluding contour*** or ***outline*** of the quadric. In 1998, Cross *et al.* described in [4] a linear triangulation scheme for a quadric from its outlines using dual space geometry. The important result in [4] was that, like linear triangulation for points, the proposed scheme works for both *finite* and *general projective cameras*, according to whether the camera maps a 3D Euclidean or a projective world to pixels (see terminology in [5, p157]). Nevertheless, a key difference is that triangulation for a quadric requires at least 3 outlines from 3 views whereas with only 2 outlines an ambiguity remains i.e., a 1D linear family of solutions is found.

In this work, we investigate ***Quadrics of Revolution*** (***QoR's***) which are 7-dof quadrics generated by revolving a conic about one of its symmetry axis. We describe a multiple view algorithm that *unambiguously* reconstructs a ***Prolate Quadric of Revolution*** (***PQoR***) given its occluding contours from 2 or more views taken by known finite projective cameras. The term PQoR refers to any QoR such that its revolution axis is the axis through the two real foci of the revolved conic, called ***principal foci*** of the PQoR. Albeit the term 'prolate' usually only applies to *ellipsoids of revolution*, PQoR's will also include here *hyperboloids of two sheets of revolution*.

The problem is stated using dual space geometry and our contributions can be summarized as follows.

- We describe a 7-dof parameterization of a PQoR via a 'linear combination' of two dual quadrics: the *quadric dual to the principal focus-pair* of the PQoR and the *dual absolute quadric*,
- We prove that the images of the principal foci can be recovered set-wise from the image of the PQoR and the image of the dual absolute quadric.

The ***quadric dual to the principal focus-pair*** is a (degenerate) rank-2 quadric envelope consisting of the two (real) principal foci of the PQoR. The ***dual absolute quadric*** (***DAQ*** [5, p84]) is also a (degenerate) rank-3 dual quadric, denoted $\mathtt{Q}_\infty^*$, which can be regarded as the (plane-)envelope of the ***absolute conic*** (***AC***) in 3-space. Its image coincides with the image of the dual absolute conic, denoted $\boldsymbol{\omega}^*$.

Prior works. In the earliest works, Ferry *et al.* [3] reconstructed a *known* QoR, from one outline and Ma [1] reconstructed a general ellipsoid given three outlines. Cross *et al.* [4] reconstructed quadrics given three outlines from three views or two outlines with one matched point while Shashua *et al.* [9] requires one outline and four matched points. Eventually, Wijewickrema *et al.* [7] described a two-step algorithm for reconstructing spheres given two outlines from two views. Up to our knowledge, there is no existing work that describes how to linearly reconstruct a quadric of revolution from two views or more using a linear method.

Notations. All vectors or matrices are homogeneous i.e., they represent points or homographies in nD-projective space, hence their lengths or orders are $n+1$.

We will basically follow the same notations as in [5]. For any square matrix $\mathtt{A}$, $\mathrm{eig}(\mathtt{A})$ refers to the set of eigenvalues of the matrix $\mathtt{A}$ and $\mathrm{eig}(\mathtt{A},\mathtt{B})$ refers to the set of generalized eigenvalues of the matrix-pair $(\mathtt{A},\mathtt{B})$.

Finite projective cameras are described by 3×4 matrices $\mathtt{P}^i = \mathtt{K}^i\mathtt{R}^i\left[\,\mathtt{I} \mid \mathbf{t}^i\,\right]$, where $\mathtt{R}^i$ and $\mathbf{t}^i$ relate the orientation and position of camera i to some Euclidean 3D-world coordinate system. The matrix $\mathtt{K}^i$ is the calibration matrix (of *intrinsics*) which has the general form given in [5, Eq. (6.10), p157].

1.1 Background

Projective quadrics. In order-n projective space $\mathbb{P}_n$, a quadric is the locus of points $\mathbf{X} \in \mathbb{P}_n$ satisfying a quadratic equation $\mathbf{X}^\top \mathtt{Q}\mathbf{X} = 0$ for some order-$(n+1)$ symmetric matrix $\mathtt{Q}$, called *quadric matrix.* If $n = 2$ a quadric is called a conic. In dual projective space $\mathbb{P}_n^*$, a quadric is the envelope of hyperplanes $\boldsymbol{\pi} \in \mathbb{P}_n^*$ satisfying a quadratic equation $\boldsymbol{\pi}^\top \mathtt{Q}^* \boldsymbol{\pi} = 0$, for some 'dual' order-$(n+1)$ symmetric matrix $\mathtt{Q}^*$. A proper quadric is a self-dual surface, comprising both a quadric locus and a quadric envelope [11, p267] as $\mathtt{Q} \sim (\mathtt{Q}^*)^{-1}$ if the hyperplanes of $\mathtt{Q}^*$ are exactly the tangent hyperplanes of $\mathtt{Q}$. In this work, to be consistent with [5, p73], we will also use the term 'dual quadric' when referring to a quadric envelope.

Under any homography $\mathtt{H}$, a quadric locus $\mathtt{Q}$ maps to $\mathtt{Q}' \sim \mathtt{H}^{-\top}\mathtt{Q}\mathtt{H}^{-1}$, while the envelope $\mathtt{Q}^*$ maps to $\mathtt{Q}'^* \sim \mathtt{H}\mathtt{Q}^*\mathtt{H}^\top$. The action of a camera $\mathtt{P}$ maps a quadric $\mathtt{Q}$ to conic $\mathtt{C}$ on the image plane [5, p201] and this is derived in dual form by $\mathtt{C}^* \sim \mathtt{P}\mathtt{Q}^*\mathtt{P}^\top$.

The *rank* of a quadric (in locus or envelope form) is the rank of its matrix and a rank-deficient quadric is said to be degenerate. What is important to be reminded is that a degenerate quadric envelope $\mathtt{Q}^*$ with rank-2 consists of a pair of points $(\mathbf{Y}, \mathbf{Z})$ as it can be written as $\mathtt{Q}^* = \mathbf{Y}\mathbf{Z}^\top + \mathbf{Z}\mathbf{Y}^\top$. We will often refer to the envelope $\mathtt{Q}^*$ as the ***quadric dual to the point-pair*** $(\mathbf{Y}, \mathbf{Z})$.

Signature of projective quadrics. A projective quadric (in locus or envelope form), whose matrix is real, has a different type according to its *signature*. The signature is defined as (ξ_1, ξ_2), where ξ_1 and ξ_2 are the following functions of its matrix eigenvalues

$$\xi_1 = \max(\rho, \nu) \text{ and } \ \xi_2 = \min(\rho, \nu),$$

in which ρ and ν respectively count the positive and negative eigenvalues. Thanks to Sylvester's law of inertia [12, p. 403], the signature is projectively invariant in the sense of being the same in any projective representation.

In particular the signature of any rank-2 quadric envelope $\mathtt{Q}^*$ is

$$(\xi_1, \xi_2) = \begin{cases} (1,1) \text{ if } \mathtt{Q}^* \text{ consists of a pair of } \textit{distinct real points}, \\ (2,0) \text{ if } \mathtt{Q}^* \text{ consists of a pair of } \textit{conjugate complex points}. \end{cases} \tag{1}$$

Note that $\xi_1 + \xi_2 = \operatorname{rank} \mathtt{Q}^*$ so the rank is also projectively invariant.

2 Parameterization and Reconstruction of a Prolate Quadric of Revolution (PQoR)

2.1 A dual parameterization of a PQoR

We describe here the proposed parameterization of a PQoR whose matrix is denoted $\mathtt{Q}$ in the sequel.

Proposition 1. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be the* 4*-vectors of two distinct real points in 3-space and let* $\mathtt{Q}^*_\infty$ *be the matrix of the DAQ.*

The set of PQoR's having $\mathbf{F}$ *and* $\mathbf{G}$ *as principal foci can be represented, in envelope form, by* 4×4 *generic symmetric matrices*

$$\mathtt{Q}^*(u) = u\mathtt{X}^* + (1-u)\mathtt{Q}^*_\infty, \tag{2}$$

where $u \in \mathbb{R}$ *is a free variable and*

$$\mathtt{X}^* = \mathbf{F}\mathbf{G}^\top + \mathbf{G}\mathbf{F}^\top,$$

is the quadric dual to the principal focus-pair $(\mathbf{F}, \mathbf{G})$*.*

This proposition actually makes use of a known result of projective geometry so its proof can be found elsewhere (e.g., see [11, 13]). Just one remark about it. The set of matrices of the above proposition represents a 1D linear family of quadrics in envelope form, such a family being called a ***range of quadrics***[1] in [11, p335], which is the dual of a pencil of quadrics.

Our key idea is to write $\mathtt{Q}$, the PQoR to be reconstructed, in dual form as

$$\mathtt{Q}^* = \mathtt{X}^* - x_0\mathtt{Q}^*_\infty, \tag{3}$$

where x_0 is a nonzero scalar, that we will call the ***projective parameter*** of $\mathtt{Q}^*$.

A minimal parameterization. It is well-known that the DAQ is fixed under similarities and has the canonical form

$$\mathtt{Q}^*_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in any Euclidean representation [5, p84]. Hence, the matrix form (3) is a dual Euclidean 7-dof parameterization of a general PQoR since the rank-2 matrix $\mathtt{X}^*$ clearly has indeed six dof's and the scalar x_0 has one. This is consistent with existing Euclidean 'locus' parameterizations e.g., that given in [14].

2.2 A dual parameterization of the image of a PQoR.

Any range of quadrics projects to a range of conics (i.e., to a 1D linear family of conics envelopes) whose members are the images of the quadrics of the range.

Next a proposition that can be seen as the corollary of proposition 1.

[1] A necessary and sufficient condition for a quadric range to consist entirely of envelopes of confocal quadrics is that it includes $\mathtt{Q}^*_\infty$ [11, p335]. A quadric range satisfying this condition is called a ***confocal range***.

Corollary 1. *The image of the range of quadrics (2) is the range of conics represented by* 3×3 *generic symmetric matrices*

$$\mathtt{C}^*(v) = v\mathtt{Y}^* + (1-v)\boldsymbol{\omega}^*, \tag{4}$$

where $v \in \mathbb{R}$ *is a free variable and*

$$\mathtt{Y}^* = \mathbf{f}\mathbf{g}^\top + \mathbf{g}\mathbf{f}^\top, \tag{5}$$

is the conic dual to the images $\mathbf{f}, \mathbf{g}$ *of principal foci* $\mathbf{F}, \mathbf{G}$ *and*

$$\boldsymbol{\omega}^* = \mathtt{K}\mathtt{K}^\top$$

is the dual image of the AC.

There is no difficulty with the proof of this corollary which is hence omitted.

Why is the corollary important? The range of conics (4), image of the quadric range (2), is a 1D linear family of envelopes including $\mathtt{C}^*$ i.e., the dual occluding contour of the PQoR. As any conic range, it can be spanned by any two of its members, in particular by $\mathtt{C}^*$ and $\boldsymbol{\omega}^*$ so we can also refer to it as $\mathtt{C}^* - y\boldsymbol{\omega}^*$. The conic range $\mathtt{C}^* - y\boldsymbol{\omega}^*$ includes three degenerate members with singular matrices $\mathtt{C}^* - \lambda_i\boldsymbol{\omega}^*$, $i \in \{1..3\}$, among which is the dual conic (5).

Now our key result: proposition 2 claims that the envelope (5) can be uniquely identified among the three degenerate members of $\mathtt{C}^* - y\boldsymbol{\omega}^*$, thanks to its (projectively invariant) signature.

Proposition 2. *Given the image* $\boldsymbol{\omega}$ *of the AC and the occluding contour* $\mathtt{C}$ *of a PQoR, the images* $\{\mathbf{f}, \mathbf{g}\}$ *of the real principal foci of the PQoR can be set-wise recovered i.e., the degenerate dual conic (5) can be uniquely recovered.*

Proof. The conic range $\mathtt{C}^* - y\boldsymbol{\omega}^*$ includes three degenerate envelopes with singular matrices $\mathtt{C}^* - \lambda_i\boldsymbol{\omega}^*$ where λ_i is a generalized eigenvalue of the matrix-pair $(\mathtt{C}^*, \boldsymbol{\omega}^*)$ i.e., a root of the characteristic equation $\det(\mathtt{C}^* - y\boldsymbol{\omega}^*) = 0$. Hence, there exists λ_{i_0} such that $\mathtt{Y}^* \sim \mathtt{C}^* - \lambda_{i_0}\boldsymbol{\omega}^*$. Since $\mathtt{Y}^*$ has signature $(1,1)$, cf. (1), we now just show that the two other degenerate members have signatures $(2,0)$x.

On the one hand, it is known that the generalized eigenvalues of $(\mathtt{C}^*, \boldsymbol{\omega}^*)$ are also the (ordinary) eigenvalues of the matrix $\mathtt{C}^*\boldsymbol{\omega}$, where $\boldsymbol{\omega} = (\boldsymbol{\omega}^*)^{-1}$. Furthermore if $\mathtt{H}$ is a order-3 nonsingular transformation then $\mathtt{C}^*\boldsymbol{\omega}$ and $\mathtt{H}(\mathtt{C}^*\boldsymbol{\omega})\mathtt{H}^{-1}$ are similar in the sense of having the same set of eigenvalues. Consequently,

$$\begin{aligned}\operatorname{eig}(\mathtt{C}^*, \boldsymbol{\omega}^*) &= \operatorname{eig}(\mathtt{C}^*\boldsymbol{\omega}) = \operatorname{eig}(\mathtt{H}\mathtt{C}^*\boldsymbol{\omega}\mathtt{H}^{-1}) = \operatorname{eig}(\mathtt{H}\mathtt{C}^*(\mathtt{H}^\top\mathtt{H}^{-\top})\boldsymbol{\omega}\mathtt{H}^{-1}) \\ &= \operatorname{eig}(\mathtt{H}\mathtt{C}^*\mathtt{H}^\top, \mathtt{H}\boldsymbol{\omega}^*\mathtt{H}^\top).\end{aligned}$$

This entails that the set of the projective parameters of the degenerate members of $\mathtt{C}^* - y\boldsymbol{\omega}^*$ is the same as that of $\mathtt{H}\mathtt{C}^*\mathtt{H}^\top - y\mathtt{H}\boldsymbol{\omega}^*\mathtt{H}^\top$ as it remains invariant —as a set– under any homography $\mathtt{H}$. This allows us to carry on the proof using the most convenient projective representation.

On the other hand, there exists an homography $\mathtt{H}$ of the image plane such that

$$\mathtt{HC}^*\mathtt{H}^T = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \triangleq \mathtt{D} \text{ and } \mathtt{H}\boldsymbol{\omega}^*\mathtt{H}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathtt{I}, \tag{6}$$

where $a, b, c \in \mathbb{R} \setminus \{0\}$ assuming, without loss of any generality, that $a > b > c$. To be convinced, if $\mathtt{VDV}^\top = \bar{\mathtt{C}}^*$ is the eigen-decomposition ($\mathtt{V}$ is orthogonal) of the 'calibrated' occluding contour $\bar{\mathtt{C}}^* = \mathtt{K}^{-1}\mathtt{C}^*\mathtt{K}^{-\top}$ then $\mathtt{H} = \mathtt{V}^\top\mathtt{K}^{-1}$ is such an homography. The projective parameters λ_i of the degenerate envelopes $\mathtt{C}^* - \lambda_i\boldsymbol{\omega}^*$ can be formally computed as the generalized eigenvalues of $(\mathtt{D}, \mathtt{I})$ —since $\bar{\boldsymbol{\omega}}^* \triangleq \mathtt{I}$ represents the calibrated dual image of the AC, see (6)— i.e., as the ordinary eigenvalues of $\mathtt{D}$. Clearly these are a, b, c. Then it is straightforward to compute, in this order, the matrices $\bar{\mathtt{Y}}^{*i} = \mathtt{D} - \lambda_i\bar{\boldsymbol{\omega}}^*$ and their (ordinary) eigenvalues:

$$\bar{\mathtt{Y}}^{*1} = \begin{pmatrix} a-c & 0 & 0 \\ 0 & b-c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{eig}(\bar{\mathtt{Y}}^{*1}) = \begin{Bmatrix} a-c \\ b-c \\ 0 \end{Bmatrix}, \quad (\xi_1^1, \xi_2^1) = (2, 0)\,,$$

$$\bar{\mathtt{Y}}^{*2} = \begin{pmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c-b \end{pmatrix}, \quad \mathrm{eig}(\bar{\mathtt{Y}}^{*2}) = \begin{Bmatrix} a-b \\ c-b \\ 0 \end{Bmatrix}, \quad (\xi_1^2, \xi_2^2) = (1, 1)\,,$$

$$\bar{\mathtt{Y}}^{*3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & c-a \end{pmatrix}, \quad \mathrm{eig}(\bar{\mathtt{Y}}^{*3}) = \begin{Bmatrix} b-a \\ c-a \\ 0 \end{Bmatrix}, \quad (\xi_1^3, \xi_2^3) = (2, 0)\,.$$

Since the signatures are projectively invariant, the proof is ended. ■

The proof actually showed that $\mathtt{Y}^*$, the conic dual to the image of the principal focus-pair as given in (5), is the only degenerate member of $\mathtt{C}^* - y\boldsymbol{\omega}^*$ with signature $(1, 1)$, cf. (1). Algorithm 1 details all the steps based on the proof.

Algorithm 1 *Input* $\mathtt{C}$: PQoR's occluding contour; $\mathtt{K}$: intrinsics. *Output* $\mathbf{r}$: image of the revolution axis; $\mathbf{fg}^\top + \mathbf{gf}^\top$: conic dual to the images of the principal foci.

1. 'Calibrate' the occluding contour $\bar{\mathtt{C}}^* = \mathtt{K}^{-1}\mathtt{C}^*\mathtt{K}^{-\top}$
2. Compute the eigenvalues and eigenvectors $\{\lambda_i, \mathbf{V}_i\}_{i=1..3}$ of $\bar{\mathtt{C}}^*$
3. Find the (unique) matrix $\bar{\mathtt{C}}^* - \lambda_{i_0}\mathtt{I}$ having signature $(1, 1)$, with $i_0 \in \{1..3\}$
4. ... The image of the revolution axis is $\mathbf{r} = \mathtt{K}\mathbf{V}_{i_0}$
5. ... The dual conic $\mathbf{fg}^\top + \mathbf{gf}^\top$ is the 'uncalibrated' dual conic $\mathtt{K}(\bar{\mathtt{C}}^* - \lambda_{i_0}\mathtt{I})\mathtt{K}^\top$

3 Reconstructing a PQoR from its occluding contours

3.1 Case of two calibrated views

Fitting a PQoR to 3D planes. Let $\mathtt{C}$ be the occluding contour of $\mathtt{Q}$. The back-projection in 3-space of any image line $\mathbf{l}$ tangent to $\mathtt{C}$ at a point $\mathbf{p}$ is the 3D plane $\boldsymbol{\pi} = \mathtt{P}^\top \mathbf{l}$ [5, p197], defined by the camera center and the pole-polar relation $\mathbf{l} = \mathtt{C}\mathbf{p}$. This plane belongs the envelope of $\mathtt{Q}$ i.e., $\boldsymbol{\pi}^\top \mathtt{Q}^* \boldsymbol{\pi} = 0$ as shown in fig. 1 (a). Hence, using the parameterization (3), the following constraint is satisfied:

$$\boldsymbol{\pi}^\top \mathtt{X}^* \boldsymbol{\pi} = x_0 \, (\boldsymbol{\pi}^\top \mathtt{Q}^*_\infty \boldsymbol{\pi}). \tag{7}$$

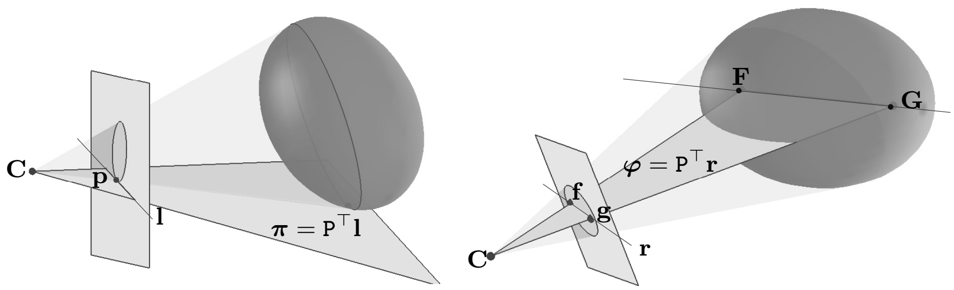

Fig. 1. (a) The plane passing through the camera center and tangent to the occluding contour is tangent to the PQoR. (b) The plane passing through the camera center and the images **f**, **g** of the principal foci **F**, **G** passes through the axis of revolution.

Constraint of the revolution axis. A direct consequence of proposition 2 is that the image of the revolution axis can be uniquely recovered.

The revolution axis is the 3D line passing through **F** and **G** in space. It projects to the line **r**, passing through their images **f** and **g**, whose back-projection in 3-space is the plane $\boldsymbol{\varphi} = \mathtt{P}^\top \mathbf{r}$ as shown in fig. 1 (b). Since the plane $\boldsymbol{\varphi}$ passes through the camera center and the revolution axis, the following constraint holds:

$$\mathtt{X}^* \boldsymbol{\varphi} = \mathbf{0}_4. \tag{8}$$

The basic reconstruction equation system. Assume that a PQoR $\mathtt{Q}$ is seen by two calibrated views and that the occluding contours $\mathtt{C}^i$ of $\mathtt{Q}$ are given in each view $i \in \{1, 2\}$. Thanks to proposition 2, we also have at our disposal the images of the principal foci of $\mathtt{Q}$, 'pair-wise packaged' in a rank-2 quadric envelope $\mathtt{Y}^{*i}$.

Let $\mathbf{l}^i_j$, $j = 1..2$, be two image lines tangent to $\mathtt{C}$ and denote $\mathbf{r}^i$ the image of the revolution axis i.e., of the line through the images of the principal foci. Hence, for each view, we have two equations (7) and one equation (8), which yields as a whole ten independent linear equations on the unknown 11-vector

$$\mathbf{X} = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})^\top, \tag{9}$$

providing we write (the symbol '$*$' denotes the corresponding symmetric element)

$$\mathtt{X}^* = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ * & x_5 & x_6 & x_7 \\ * & * & x_8 & x_9 \\ * & * & * & x_{10} \end{pmatrix}. \tag{10}$$

We can rewrite the set of equations (7,8) in the matrix form

$$\begin{pmatrix} \mathtt{A}^1 \\ \mathtt{A}^2 \end{pmatrix} \mathbf{X} = \mathbf{0}_{10}, \tag{11}$$

where $\mathtt{A}^i$ is the 5×11 data matrix associated with view i:

$$\mathtt{A}^i = \begin{bmatrix} (a_1^i)^2{+}(b_1^i)^2{+}(c_1^i)^2 & (a_1^i)^2 & 2a_1^i b_1^i & 2a_1^i c_1^i & 2a_1^i d_1^i & (b_1^i)^2 & 2b_1^i c_1^i & 2b_1^i d_1^i & (c_1^i)^2 & 2c_1^i d_1^i & (d_1^i)^2 \\ (a_2^i)^2{+}(b_2^i)^2{+}(c_2^i)^2 & (a_2^i)^2 & 2a_2^i b_2^i & 2a_2^i c_2^i & 2a_2^i d_2^i & (b_2^i)^2 & 2b_2^i c_2^i & 2b_2^i d_2^i & (c_2^i)^2 & 2c_2^i d_2^i & (d_2^i)^2 \\ 0 & 0 & 0 & \varphi_1^i & 0 & 0 & \varphi_2^i & 0 & \varphi_3^i & \varphi_4^i & 0 \\ 0 & 0 & \varphi_1^i & 0 & 0 & \varphi_2^i & \varphi_3^i & \varphi_4^i & 0 & 0 & 0 \\ 0 & \varphi_1^i & \varphi_2^i & \varphi_3^i & \varphi_4^i & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

using the notation $\boldsymbol{\pi}_j^i = (a_j^i, b_j^i, c_j^i, d_j^i)^\top$.

System (11) is the *basic reconstruction equation system.* The last three rows of $\mathtt{A}^i$ correspond to constraint (8) substituting $\boldsymbol{\varphi}^i = \mathtt{P}^{i\top}\mathbf{r}^i$ for $\boldsymbol{\varphi}$. They fix 6 out of the 10 dof's of $\mathbf{X}$ in (9). Note that, for $i = 1..2$, these constraints ensure that $\operatorname{rank} \mathtt{X}^* = 2$. The first two rows of $\mathtt{A}^i$ correspond to two constraints (7), substituting $\boldsymbol{\pi}_j^i = \mathtt{P}^{i\top}\mathbf{l}_j^i$ for $\boldsymbol{\pi}$ with $j = 1..2$; they fix the remaining 4 dof's. System (11) is now a consistent system i.e., $\operatorname{rank} \begin{pmatrix} \mathtt{A}^1 \\ \mathtt{A}^2 \end{pmatrix} = 10$ so a non-trivial *exact* solution for $\mathbf{X}$ (e.g., under the constraint $\|\mathbf{X}\|^2 = 1$) exists. In other words, the problem of the reconstruction of a quadric of revolution from its occluding contours in two calibrated views is well-posed.

3.2 Case of multiple calibrated views

In practice, multiple calibrated views, say $n \geq 2$, can be available and the system (11) will transform to $\mathtt{D}\mathbf{X} \approx \mathbf{0}_{5n}$, where $\mathtt{D}$ is a $5n \times 11$ *design matrix* that stacks all the $\mathtt{A}^i$'s, $i = 1..n$, and the operator '$\approx$' expresses the fact that data matrices $\mathtt{A}^i$ are generally perturbed by noise. Numerically speaking, we will seek a closed-form solution to the total least-squares problem $\min_{\mathbf{X}} \|\mathtt{D}\mathbf{X}\|^2$ s.t. $\|\mathbf{X}\| = 1$.

Algorithm 2 details all the steps for constructing the design matrix $\mathtt{D}$. The solution can be taken as the singular vector of $\mathtt{D}$ associated with the smaller singular value [5, p593]. Note that, in this case, we will ignore the constraint $\operatorname{rank} \mathtt{X}^* = 2$.

Algorithm 2 *Input* $\{\mathtt{P}^i\}$: finite projective cameras; $\{\mathtt{C}^i\}$: occluding contours; $\{\mathtt{K}^i\}$: intrinsics. *Output* $\mathtt{D}$: the design matrix for reconstructing the PQoR.

For each view i

1. Pick up two points $\mathbf{p}_j^i$ of $\mathtt{C}^i$, with $j \in \{1..2\}$
2. Let $\mathbf{l}_j^i = \mathtt{C}^i \mathbf{p}_j^i$ be the tangent lines at $\mathbf{p}_j^i$
3. Let $\boldsymbol{\pi}_j^i = \mathtt{P}^{i\top} \mathbf{l}_j^i$ be the back-projections of $\mathbf{l}_j^i$
4. Using algorithm 1, recover $\mathbf{r}^i$, the image of the revolution axis
5. Let $\boldsymbol{\varphi}^i = \mathtt{P}^{i\top} \mathbf{r}^i$ be the back-projections of $\mathbf{r}^i$
6. Stack the block-matrix $\mathtt{A}^i$, as defined in (12), into the design matrix $\mathtt{D}$.
7. Solve $\min_{\mathbf{X}} \|\mathtt{D}\mathbf{X}\|^2$ s.t. $\|\mathbf{X}\| = 1$

4 Results

4.1 Synthetic data

In this section, we assess the performance of the proposed algorithm by carrying out comparison tests with Cross' algorithm described in [15] using synthetic data. First, we remind the reader that only the proposed method can reconstruct a PQoR from two views. Cross' algorithm returns in such a minimal case a 1D linear family of solutions i.e., a quadric range of solutions.

In the first experiment, simulations have been achieved using a set of three calibrated cameras, with varying positions and orientations. Cameras are randomly and approximatively placed at a constant distance from the scene origin. Each camera fixates a random point located in a sphere of varying radius centred at the origin. Ellipsoids of revolution are randomly created with its center varying within this sphere. We compute the exact images of the ellipsoid in all views and then add Gaussian noise to the points of the obtained occluding contours, with zero mean and standard deviation $\sigma = n\%$ of the major axis of the image, where n varies between 0 and 5. We compute different types of errors between the exact and the reconstructed quadrics: relative errors on the minor and the major axes, the absolute errors between the centers, errors on the angles between the directions of the axes of rotation and eventually errors on the volumes. For each level of n, 500 independent trials were performed using our proposed approach and that of Cross [14]. The results are presented (Figure 2, page 10).

We can note that the errors linearly increase with regard to the noise level. We can also see that our method has clearly better performances than Cross' method.

In the second experiment, we investigate the importance of the number of views for the reconstruction. We run the same experiments but now varying the number of cameras from 2 to 8. All results are averages of 500 independent trials using a Gaussian noise with $\sigma = 2\%$ of the major axis of the image of the quadric (Figure 3, page 11). The results for Cross' approach start from three views only. Due to the fast increasing number of constraints, this second experiment shows

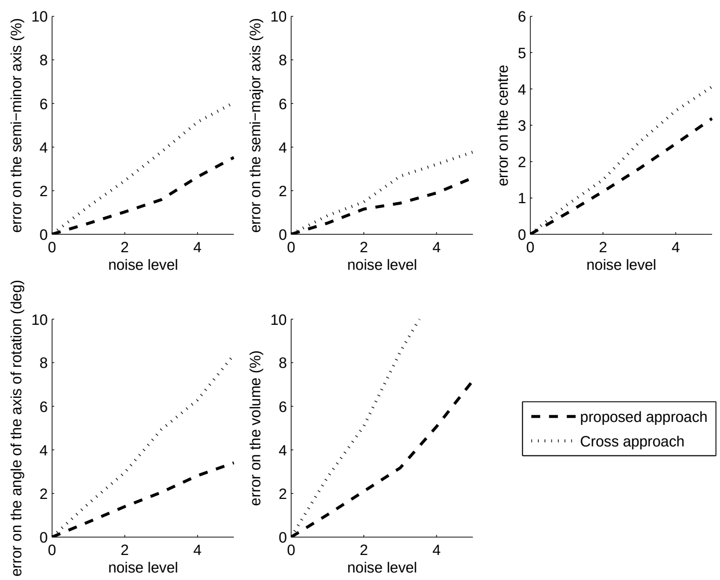

Fig. 2. The proposed algorithm are compared to Cross' one (see text). Three cameras are used for this experiment (Cross' algorithm requires at least three cameras).

that the errors decrease exponentially as the number of views increases. We can again note that our method clearly is the most accurate.

4.2 Real data

An application perfectly suited to illustrate our work is the reconstruction of objects like fruits and, in particular, bunch of grapes, where grapes are modelled by prolate ellipsoids of revolution. We used two calibrated cameras, looking at the grape with an angle of approximatively 30 degrees ; the obtained result is shown in figure 4.

5 Conclusions

In this work, we describe a multiple view algorithm that unambiguously reconstructs so-called prolate quadrics of revolution, given at least two finite projective cameras. In particular, we show how to obtain a closed-form solution. The key result was to describe how to recover the conic dual to the images of the principal foci of the quadric, providing the calibration matrix is known. Albeit not described in this paper, it is possible to recover the two points individually, even if it is not possible to distinguish one from the other. A future work is to compute the epipolar geometry of the scene i.e., the essential matrix, from the images of the principal foci in the two views, using a RANSAC-like algorithm, and hence only requiring the calibration matrix for reconstructing PQoR's.

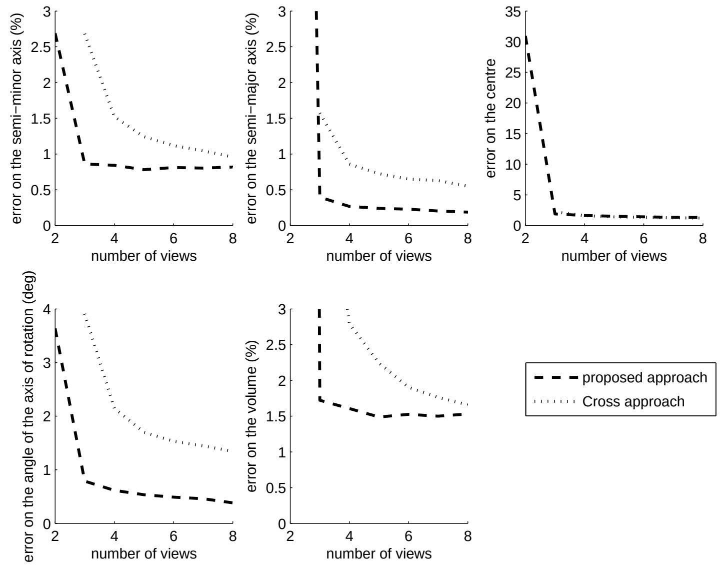

Fig. 3. Comparison tests (see text). Algorithms are compared with respect to the number of cameras.

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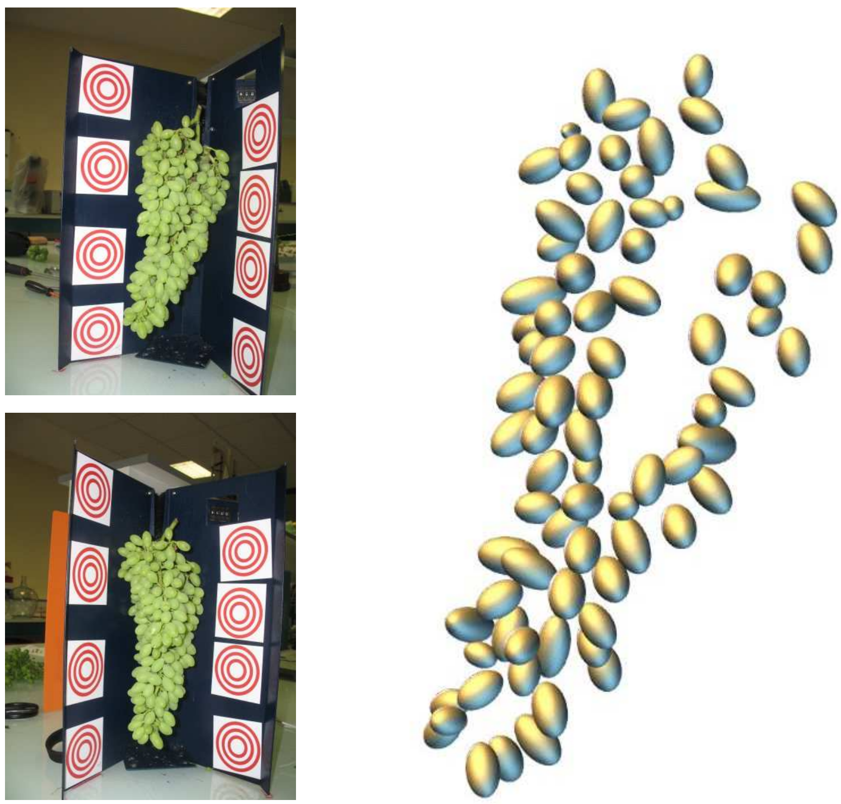

Fig. 4. Reconstruction of a bunch of grapes (see text). This is an interactive graphics: click to activate.